Mars Gravity Field from Satellite-to-Satellite Doppler Data

A. Vijayaraghavan

Jet Propulsion Laboratory
California Institute of Technology
4800 Oak Grove Drive
Pasadena, California 91109.

Introduction

The Mars Global Surveyor and the Mars Pathfinder spacecrafts are scheduled to be launched by the end of 1996. Further exploration of Mars with low-cost missions will probably continue over the next decade or two, taking advantage of launch opportunities available every two years. in this context, two or more satellites are likely to be simultaneously orbiting Mars during some overlapping period of their lifetime. In Reference 1, it has been shown by detailed covariance analysis (of a few cases) that satellite-to-satellite (STS) Doppler data is very useful in the accurate determination of the Martian gravity field. In this paper, an approximate analysis will be presented on the improvement to be obtained in the high-frequency or short wavelength Martian gravity field with STS Doppler data, avoiding costly, time-consuming and computation-intensive covariance analysis. With the present emphasis on on-board and autonomous navigation, STS Doppler data may become a reality in the not so distant future,

Satellite-to-satellite Doppler data can be obtained in two different configurations of the two spacecrafts involved. (In this abstract, sometimes Satellite-to-satellite Doppler data will simply be designated as STS data, for convenience.) A Communications-Relay cum Navigation Satellite may be deployed in a high orbit (of radius possibly 1.5,000-30,000 km) about Mars and the other in a low orbit at an altitude of about 180 km. This case will be referred to as the high-low satellite configuration. Otherwise, two spacecraft; in low orbits such as for high-resolution imaging purposes, or atmospheric studies, may be considered for STS data. The latter will be designated as the low-low satellite configuration, Both these cases are examined in the analysis below and the detailed results will be presented in the paper.

Some Preliminaries

The analysis is based on Hill's variational equations for circular orbits. The perturbations are considered to be derived from gravitational harmonics only. In particular, the probelm is solved using two-dimensional Fourier transforms in Cartesian coordinates (for the upper half-space) for the conservative gravitational field. Due to the assumptions made in the analysis, the results must be considered appropriate for high-frequency or short wavelength harmonics only.

The S1'S data consists of the relative velocity between the two satellites under consideration. For satellites in the high-low configuration, the satellite in high orbit is essentially unaffected by the higher degree and order gravitational harmonics except the fundamental, spherically symmetric field. Hence tile relative velocity between the spacecraft can be attributed to the high frequency gravitational field only. In the low-low configuration, for two satellites in the same low circular orbit, but separated by a finite distance (separated in true anomaly) between them, the relative velocity is obtained from the change in the non-spherical gravitational field due to the difference between the spacecraft positions.

The relative velocit y between the two spacecraft s is derived in the (Fourier) transform domain in terms of the high-frequency (or non-spherical part of the) gravitational field by solving Hill's equations. The perturbation forces in Hill's equations are also expressed in terms of the two-dimensional Fourier transforms of the anomalous (or spherically asymmetric part of the) gravitational field for these purposes. With an optimal filter in the frequency domain, the improvement in the spatial power spectral density of the gravity field is evaluated from the relative velocity measurements or STS (Doppler) data, This procedure primarily completes the analysis.

Just the results from the crucial steps of the analysis are presented in this abstract for both the high-low and low-low satellite configurations, along with a short note on the final results. All the details will be included in the paper.

Analysis

Hill's equations[2] for the perturbations of a spacecraft nominally in a circular orbit are given by

$$\ddot{\xi} - 2n\dot{\eta} - 3n^2\xi = f_\xi \tag{1}$$
$$\ddot{\eta} + 2n\dot{\xi} = f_\eta \tag{2}$$
$$\ddot{\zeta} + n^2\zeta = f_\zeta \tag{3}$$

where (ξ, η, ζ) are the perturbations in the spacecraft position in the radial, down-track and cross-track directions respectively. $\dot{\xi}$ and $\ddot{\xi}$ denote the velocity and acceleı ation in the radial direction and similarly $(\dot{\eta}, \ddot{\eta})$ and $(\dot{\zeta}, \ddot{\zeta})$ in the down-track and cross-track directions. n denotes the mean motion of the spacecraft in its nominal circular orbit an d (n = V_0/it), where V_0 and R arc the nominal circular speed and orbital radius.

The most **crucial assumption** in the approximate analysis **is that the planetary surface shall be considered '{flat". in** particular, let the Cartesian $(x\text{-}y)$ plane denote the planetary surface with the x axis parallel to the nominal down-track motion of the spacecraft and the y axis parallel to the cross-track direction and pointing in the same manner. The z axis points "radially upward", in the upper half-space.

With this nomenclature, it is readily seen that

$$\dot{\xi} = (d\xi/dx)\,(dx/dt) = V_0\,(d\xi/dx) \tag{4}$$

and similarly, $\ddot{\xi} = V_0^2(d^2\xi/dx^2)$. In turn, the perturbation equations (1 -3) can be rewritten as in

$$V_0^2\,\xi'' - 2nV_0\eta' - 3n^2\xi = f_\xi \tag{5}$$
$$V_0^2\,\eta'' + 2nV_0\zeta' = f_\eta \tag{6}$$
$$V_0^2\,\zeta'' + n^2\zeta = f_\zeta \tag{7}$$

where (') and (") denote the first and second derivatives with respect to x. The two-dimensional Fourier transforms in (x, y) of equations 5 through 7 are readily obtained:

$$-(\omega_x^2 V_0^2 + 3n^2)\tilde{\xi} - 2nV_0 j\omega_x \tilde{\eta} = \hat{f}_\xi \tag{8}$$

$$2nV_0 j\omega_x \tilde{\xi} - \omega_x^2 V_0^2 \tilde{\eta} = \hat{f}_\eta \tag{9}$$

$$-\omega_x^2 V_0^2 \tilde{\zeta} + n^2 \tilde{\zeta} = \hat{f}_\zeta \tag{10}$$

where the Fourier transform of any function $G(x, y; z)$ has been implicitly defined as in

$$\tilde{G}(\omega_x, \omega_y; z) = \iint G(x, y; z) \exp\{-j(\omega_x x + \omega_y y)\}\, dx\, dy. \tag{11}$$

The perturbation forces (f_ξ, f_η, f_ζ) and their transforms are derived from the scalar anomalous gravitational potential $G(x, y; z)$ and its transform, $\hat{G}(\omega_x, \omega_y; z)$. in particular, $G(x, y; z)$ satisfies Laplaces's equation, with prescribed values on the surface, z = O;

$$\nabla^2 G = 0; G(x, y; 0) = G(x, Y). \tag{12}$$

It is easily derived[3,4] that

$$\tilde{G}(\omega_x, \omega_y; z) = \tilde{G}(\omega)\, e^{-|\omega| z}, \; (W^2 = \omega_x^2 + \omega_y^2) \tag{13}$$

where $\tilde{G}(\omega)$ and $G(x, y, ; 0)$ are Fourier transform pairs, Since the perturbation forces are obtained from the (negative of the) gradient of the anomalous gravitational potential,

$$(f_\xi, f_\eta, f_\zeta) = -(\partial/\partial z, \partial/\partial y, \partial/\partial x)\, G(x, y, ; z) \tag{14}$$

$$(\tilde{f}_\xi, \tilde{f}_\eta, \tilde{f}_\zeta) = (|\omega|, -j\omega_x, -j\omega_y)\, \tilde{G}(\omega) e^{-|\omega| z}. \tag{15}$$

Substituting (15) in equations (8) through (10), solutions for the perturbations in the spacecraft position are obtained in the transform domain as follows:

$$\tilde{\xi} = -\{|\omega|/(\omega_x^2 V_0^2)\}\, \tilde{G}(\omega)\, e^{-|\omega| h} \tag{16}$$

$$\tilde{\eta} = \{j\omega_x/(\omega_x^2 V_0^2)\}\, \tilde{G}(\omega)\, e^{-|\omega| h} \tag{17}$$

$$\tilde{\zeta} = \{j\omega_y/(\omega_x^2 V_0^2)\}\, \tilde{G}(\omega)\, e^{-|\omega| h} \tag{18}$$

where, the spacecraft nominal altitude, 'z = h', **above** the planetary surface has specifically been entered in the equations and it has also been assumed that $R\omega_x \gg 1$.

Since the relative velocity in the down-track direction, $v_x = (d\eta/dt) = V_0 (d\eta/dx)$, its transform is given by (from the definition in equation 11 and Eqn. 17)

$$\tilde{v}_x = (j\omega_x) V_0\, \tilde{\eta} = -(1/V_0)\tilde{G}(\omega)\, e^{-|\omega| h}. \tag{19}$$

Similarly, it is easily shown that

$$\tilde{v}_y = (j\omega_x) V_0 \tilde{\zeta} = -(\omega_y/\omega_x)(1/V_0)\tilde{G}(\omega)\, e^{-|\omega| h} \tag{20}$$

$$\tilde{v}_z = (j\omega_x) V_0 \tilde{\xi} = (|\omega|/j\omega_x)(1/V_0)\tilde{G}(\omega) e^{-|\omega| h}. \tag{21}$$

(19), (20) and (21) arc the **measurement equations of the Doppler data on the relative velocity between two satellites in the high-low** configuration.

The relative velocity v_{ll}, between 2 satellites separated by a distance A in the same low circular orbit (low-low configuration) is given by[5]

$$\begin{aligned} v_{ll} &= v_x(x + \Delta/2) - v_x(x - \Delta/2) \\ &= \Delta \frac{d}{dx}\{V_0 \frac{d\eta}{dx}\} = V_0 \Delta \frac{d^2\eta}{dx^2} \end{aligned}$$

so that in the transform dom ain

$$\tilde{v}_{ll} = -(2/V_0) j \sin(\omega_x \Delta/2)\, \tilde{G}(\omega)\, e^{-|\omega| h} \tag{22}$$

In this paper, the *gaussian-weighted average of the pointwise or local surface gravity anomaly* is examined for evaluating the merits of determining the nigh-frequency gravity field with satellite-to-satellite Doppler data. Let $\lambda(x, y)$ ancl $\lambda_{av}(x, y)$ denote the pointwise surface gravity anomaly and its gaussian-weighted average; they are given by

$$\lambda(x,y) = -(\partial/\partial z)\, G(x,y;z) \text{ at } z = 0 \tag{23}$$

$$\lambda_{av}(x, y) = \frac{1}{2\pi\sigma^2} \iint \lambda(p, q) \exp\{-\frac{1}{2\sigma^2}[(x-p)^2 + (y-q)^2]\}\, dp\, dq \tag{24}$$

where σ is the appropriately chosen[4] 'spread' of the gaussian weighting kernel. It may be noted that the 'pointwise surface gravity anomaly' is simply the radial or z-directional acceleration f_ξ on the planetary surface (at $z = 0$) as in Eqn. (14).

Let $\hat{\lambda}_{av}(x, y)$ be the **"optimally estimated"** gaussian-averaged pointwise surface gravity anomaly and $\tilde{\lambda}_{av}$ denote the error as in

$$\tilde{\lambda}_{av}(x, y) = \hat{\lambda}_{av}(x,y) - \lambda_{av}(x,y) \tag{25}$$

Then the merits of determining the high-frequency gravity field with data from satellites in the "high-low" and "low-low" configuration will be evaluated by the minimum value(s) of the square-error integral in the estimated surface gravity anomaly (gaussian-averaged) as in

$$\lambda^* = \text{Min } E\{[\tilde{\lambda}_{av}(x,y)]^2\} \tag{26}$$

$$= \iint \{\hat{\lambda}_{av}(x, y) - \lambda_{av}(x,y)\}^2\, dx\, dy \tag{27}$$

By definition (ant] choice) the **optimal estimator will** yield the minimum square-error integral in equations (26) and (27). In particular, it follows from Parseval's theorem that

$$\lambda^* = (1/4\pi^2) \iint ||\tilde{\lambda}_{av}||^2\, d\omega_x\, d\omega_y \tag{28}$$

$$= (1/4\pi^2) \iint \{\hat{\lambda}_{av}(\omega) - \lambda_{av}(\omega)\}^2\, d\omega_x\, d\omega_y. \tag{29}$$

From now onwards, the (˜) above the argument as in $\tilde{\lambda}(\omega)$, denoting Fourier transform will be dropped for convenience; the context will make it clear, when the discussion is in the frequency (ω) domain.

Furthermore, from equations (13), (15), (23) and (25), $\lambda(\omega)$ and $\lambda_{av}(\omega)$ can be obtained as in

$$\lambda(\omega) = |\omega|\, G(\omega) \tag{30}$$

$$\lambda_{av}(\omega) = |\omega| \exp(-\sigma^2\omega^2/2)\, G(\omega) \tag{31}$$

$$= p(\omega)\, G(\omega) \tag{32}$$

$$\text{where } p(\omega) = |\omega| \exp(-\sigma^2\omega^2/2) \tag{33}$$

Let the genera] k-vector of measurements be denoted by

$$d(\omega) = H(\omega)\ G(w) + W \tag{34}$$

$d(\omega)$ is the S'1'S Doppler (observational) data. $H(\omega)$ is tile transfer function between the observations and the anomalous gravitational field $G(\omega)$, as in equations (19)- (21) for two satellites in the high-low configuration and as in (22) for low-low satellites. W is a k-vector of measurement (noise) errors. Let $\dot{\lambda}_{av}(\omega)$ be optimally determined from

$$\hat{\lambda}_{av}(\omega) = \psi^T(\omega)\ d(\omega) = \psi^T(\omega)\ [H(\omega)\ G(\omega) + W] \tag{35}$$

where $\psi(\omega)$ is a k-vector optimal estimator and the superscript $(\ )^T$, implies the transpose in matrix algebra. From (32) and (35), it is readily seen that

$$\bar{\lambda}_{av}(\omega) = \{\psi^T(\omega)\, H(\omega) - p(\omega)\}\ G(\omega) + \psi^T(\omega)\, W \tag{36}$$

$$\begin{aligned} ||\bar{\lambda}_{av}(\omega)||^2 = &\{\psi^T(-j\omega)\, H(-j\omega) - p(-j\omega)\}\ \Phi_G(\omega)\ \{H^T(j\omega)\, \psi(j\omega) - p(j\omega)\} \\ &+ \psi^T(-j\omega)\, \Phi_W(\omega)\, \psi(j\omega) \end{aligned} \tag{37}$$

$$\text{where } \Phi_G(\omega) = E\{||G(\omega)||^2\}. \tag{38}$$

Similarly, $\Phi_W(\omega)$ is the power spectral density of the measurement noise. Since the integrand in (28) is positive semi-definite, the minimum value of the integral for λ^* is attained, if the filter $\psi(\omega)$ is chosen so that the first variation $\delta\, ||\bar{\lambda}_{av}(\omega)||^2$ = O. From this condition, the optimal filter $\psi(\omega)$ is derived as in

$$\psi(\omega) = \{H(-j\omega)\, \Phi_G(\omega)\, H^T(j\omega) + \Phi_W(\omega)\}^{-1}\, H(-j\omega)\, \Phi_G(\omega)\, p(j\omega) \tag{39}$$

with necessary assumptions on data noise and the gravitational potential so that all cross-correlations vanish identically. In particular, for the optimal estimator, the minimum square-error integral λ^* (the familiar '(cost function") is given by

$$\begin{aligned} \lambda^* &= \text{Min } E\{||\bar{\lambda}_{av}(\omega)||^2\} \\ &= \frac{1}{4\pi^2} \iint \frac{\Phi_G(\omega)\, p(-j\omega)\, p(j\omega)}{1 + \Phi_G(\omega)\, H^T(j\omega)\, \Phi_W^{-1}\, H(-j\omega)}\, d\omega_x\, d\omega_y. \end{aligned} \tag{40}$$

Proceeding from Hill's equations, Doppler measurement of the relative velocity between two satellites is shown related to the anomalous gravitational potential as in equations (19) through (22). Then a minimum cost criterion is stipulated as given in Eqs. (26) through (29) in terms of the estimation error in the averaged pointwise surface gravity anomaly, squared and integrated over the 'planetary surface'. The optimal estimator is derived in (39) and the minimum value of the square-error integral is obtained in (40) depending upon the data type (for the transfer function) and the power spectral density of the data noise and of the surface gravity anomaly. This completes the analysis.

Results and Conclusion

'l'he denominator in Eqn. (40) for the square-error integral

$$D^* = \{1 + \Phi_G(\omega)\, H^T(j\omega)\, \Phi_W^{-1}\, H(-j\omega)\} \tag{41}$$

can be clearly interpreted as the ratio of the *a priori* to *a posteriori* variance and hence can be used as a measure of the effectiveness of a given data type with transfer function $H(j\omega)$ and measurement noise power spectral density Φ_W, which will be assumed constant in this paper.

For the high-low satellites, the transfer function has already been derived to be

$$H_{hl}(\omega) = -\exp(-\omega h)\,(1/V_0)\,\{1,\,(\omega_y/\omega_x),\,-j(\omega/\omega_x)\}. \tag{42}$$

The vector transfer function is direction-dependent, or in other words it is not isotropic in the $(\omega_x\text{-}\omega_y)$ plane.

For two satellites in the low-low configuration, the transfer function is a scalar, given by

$$H_{ll}(\omega) = -(2j)\exp(-\omega h)\;(1/V_0)\;\sin\,(\omega_x \Delta/2) \tag{43}$$

which is also not isotropic. in the $(\omega_x\text{-}\omega_y)$ plane.

The **reduction in the variance from the** *a priori* given by (41) can be calculated for each (spatial) frequency by transforming the integral in polar coordinates in the $(\omega_x\text{-}\omega_y)$ plane. The power spectral density of the non-spherical part of the Martian gravitational field for such purposes was obtained from the 50th degree and order field available from Konopliv[6]. Preliminary results[7] indicate that an improvement by a factor of 5 to 10 can be obtained in the gravity field (upto the 30th degree and order) with data from two satellites in the high-low configuration. An even greater reduction in the uncertainties in the anomalous gravitational field (by a factor of 20 to 50) is indicated, with Doppler data from two satellites in the low-low configuration. All the assumptions for the various cases and the details of the computations will be presented in the paper in full detail.

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